Determining Noise Levels in Blurry Image Data

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Suppose we have data of the form

$$b = h * u + \epsilon, \tag{1}$$

where u is an image we want to recover, h is a point spread function (PSF), and ϵ is a noise term. Suppose $\epsilon \sim N(0, \sigma^2 I)$, where σ is unknown. Let $\mathcal{F}$ denote the unitary discrete Fourier transform operator. Let $\hat{x} = \mathcal{F}x$ for an image x. Then

$$\hat{b} = \hat{h} \cdot \hat{u} + \hat{\epsilon}. \tag{2}$$

Proposition 1. *Suppose $\epsilon \sim N(0, \sigma^2 I)$. Then*

$$\hat{\epsilon} = |\hat{\epsilon}| e^{i\theta}, \tag{3}$$

where $|\hat{\epsilon}| \sim N(0, \sigma^2 I)$ and $\theta \sim U([-\pi, \pi))$.

This result is explained in the abstract of this article [1].

Proof. Before going into the rigorous proof, we provide some leading results. First

$$\begin{aligned} \mathbb{E}\hat{\epsilon}_k &= N^{-1/2} \mathbb{E} \sum_{j=0}^{N-1} \epsilon_j e^{-i2\pi kj/N} \\ &= N^{-1/2} \sum_{j=0}^{N-1} \mathbb{E}\epsilon_j e^{-i2\pi kj/N} = 0. \end{aligned} \tag{4}$$

Next

$$\begin{aligned} \mathbb{E}|\hat{\epsilon_k}|^2 &= N^{-1} \mathbb{E} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \epsilon_m \epsilon_n e^{-i2\pi k(m-n)/N} \\ &= N^{-1} \sum_{m=0}^{N-1} \mathbb{E}\epsilon_m^2 = \sigma^2. \end{aligned} \tag{5}$$

So the mean and variance match the claims of the proof.

In general, for a Gaussian variable X with mean μ covariance matrix σ, the characteristic function is given by

$$\phi_X(t) = \exp\left(i\mu t - \frac{t^2 \sigma^2}{2} \right),$$

and for any constants a, b and independent random variable X, Y the characteristic function for $aX + bY$ is

$$\phi_{aX+bY}(t) = \phi_X(at)\phi_y(bt).$$

Therefore, combining these two facts, the characteristic function for $X_k = Re\{\hat{\epsilon}_k\}$ is

$$\phi_{X_k}(t) = \exp\left(-\frac{t^2\sigma^2}{2N}\sum_{j=0}^{N-1}\cos^2(2\pi kj/N)\right) \tag{6}$$

Write $\cos^2$ as

$$\cos^2(2\pi kj/N) = \left(\frac{e^{i2\pi kj/N} + e^{-i2\pi kj/N}}{2}\right)^2 = \left(\frac{e^{i4\pi kj/N} + e^{-i4\pi kj/N} + 2}{4}\right).$$

Substituting this into the sum in (6) obtains

$$\sum_{j=0}^{N-1}\cos^2(2\pi kj/N) = N/2,$$

therefore the characteristic for X_k is

$$\exp\left(-\frac{t^2\sigma^2}{4}\right),$$

which is the characteristic of a mean zero normal distribution with variance $\sigma^2/2$. Repeating this result on the imaginary part ($Y_k = Im\{\epsilon_k\}$) of the Fourier coefficients with sines almost completes the proof. What we have shown is that

$$X_k \sim N(0, \sigma^2/2) \quad \text{and} \quad Y_k \sim N(0, \sigma^2/2).$$

The remainder of the proof would be to shown then that

$$\hat{\epsilon}_k = X_k + iY_k$$

satisfies the statement of the proposition. The claim about the squared magnitude is straightforward. The claim about the phase is not clear to me how to show. □

Now given b, we can estimate σ^2 in the following way. The Fourier transform $\hat{b}$ is composed of two parts $\hat{\epsilon}$ and $\hat{h} \cdot \hat{u}$. The first term is described in the previous proposition. The second term is something which has been low pass filtered by h. Therefore, what remained at the high pass regions should be dominated by $\hat{\epsilon}$. Therefore, we take $\hat{b}$, isolate the high wave numbers to be some set say S, and take the average over the squared terms in S to estimate σ^2:

$$\hat{\sigma}^2 = |S|^{-1}\sum_{k\in S}|\hat{b}_k|^2.$$

Be careful not to estimate σ by just averaging the magnitudes. If one wanted to do that, then it should be done with the following formula:

$$\hat{\sigma} = |S|^{-1}\sqrt{\frac{\pi}{2}}\sum_{k\in S}|\hat{b}_k|.$$

The scaling factor comes from the fact that for a random variable $X \sim N(0, \sigma^2)$, it can be shown that

$$\mathbb{E}|X| = \sqrt{\frac{2}{\pi}}\sigma.$$

Median Absolute Deviation

A more robust σ estimation is given by the median absolute deviation (MAD):

$$\hat{\sigma} = \frac{1}{0.6745} \text{median}_{k\in S} |\hat{b}_k|.$$

The median estimation is more robust to outliers, and for a normally distributed data set $\{x_i\}_i$ with mean 0 and variance 1, 50% of the distribution is on the interval $[-.6745, .6745]$ (i.e. $\phi(.6745) - \phi(-.6745) \approx .5$), hence

$$\mathbb{E}[\text{median}_i |x_i|] = .6745.$$

Fourier transform of Gaussian

This seems like a good time for a formal proof of the well known fact: *the Fourier transform of a Gaussian is a Gaussian.* This is seen in the earlier derivation, where the characteristic function is a Gaussian. I wanted to give the formal proof here:

Proposition 2. *Let*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Then

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i2\pi x\xi}\, dx = \exp\left(-2\sigma^2\pi^2\xi^2\right)$$

Proof. The key to the proof is the completing the square with a complex number:

$$x^2 + i4\pi\sigma^2\xi x = (x + i2\pi\sigma^2\xi)^2 + (2\pi\xi\sigma^2)^2. \tag{7}$$

Using this we obtain

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\sigma^2} - i2\pi x\xi\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 + i4\pi\sigma^2 x\xi\right)\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}\left((x + i2\pi\sigma^2\xi)^2 + (2\pi\xi\sigma^2)^2\right)\right) dx \\
&= \frac{\exp(-2\sigma^2\pi^2\xi^2)}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}(x + i2\pi\sigma^2\xi)^2\right) dx.
\end{aligned} \tag{8}$$

The remainder of the proof involves showing in the last line that the integral cancels the denominator. Certainly, if $\xi = 0$ this is true. It then suffices to show

$$F(a) = \int_{\mathbb{R}} \exp\left(-(x + ia)^2\right) dx$$

is a constant by showing the $F'(a) = 0$.

Note: it is probably much easier to work with $f(x) = e^{-x^2}$ and just change variables later using the scaling properties of the Fourier transform. □

References

[1] D. Freedman et al. The empirical distribution of fourier coefficients. *The Annals of Statistics*, 8(6):1244–1251, 1980.